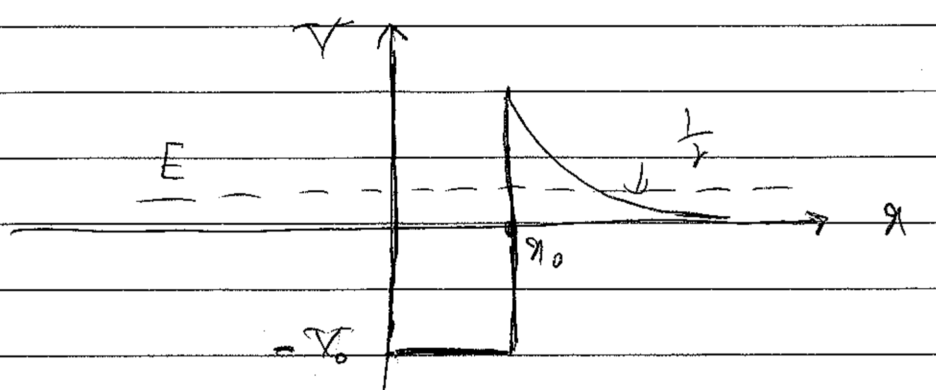
Lec 27. 10/30/2009

Tunneling:

As an important application of the WKB method, lets discuss the tunneling of α particles. The α particle (2 protons + 2 neutrons) interacts with the rest of the nucleus (N_p protons + N_n neutrons). The two contributions to the potential (assuming zero angular momentum) are from the short distance nuclear force and the Coulomb force. The former is attractive and the latter is repulsive coming from electromagnetic interaction between protons in the α particle and the nucleus.

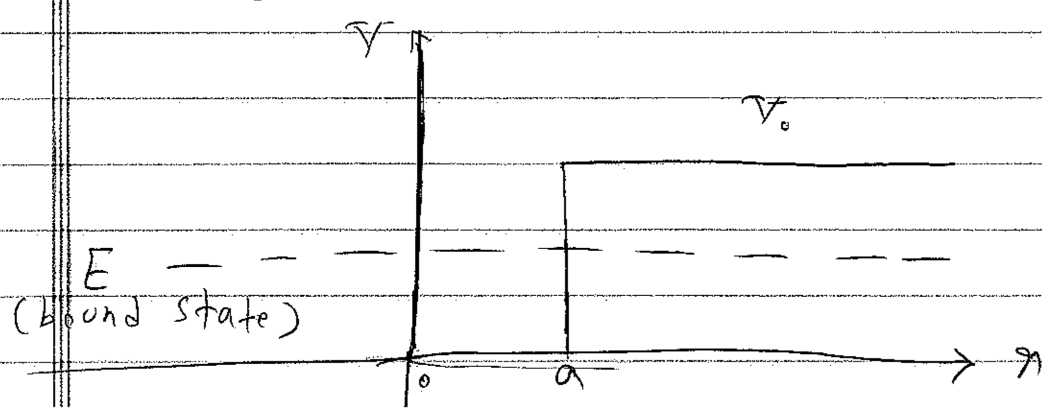
The overall potential has the following shape:



Here r is the radial distance, constant potential $-V_0$ is due to the attractive nuclear force (short distance) and the $\frac{1}{r}$ potential is from the repulsive Coulomb force.

Note that states with $E>0$ are not bound. At sufficiently large r we have $E>V$, and hence the corresponding wavefunction is oscillatory. Therefore an α particle initially confined at $r<r_0$, will eventually leak out if it has an energy $E>0$.

Now the question is that why states with positive energy should be populated. To elaborate, consider the following potential well:



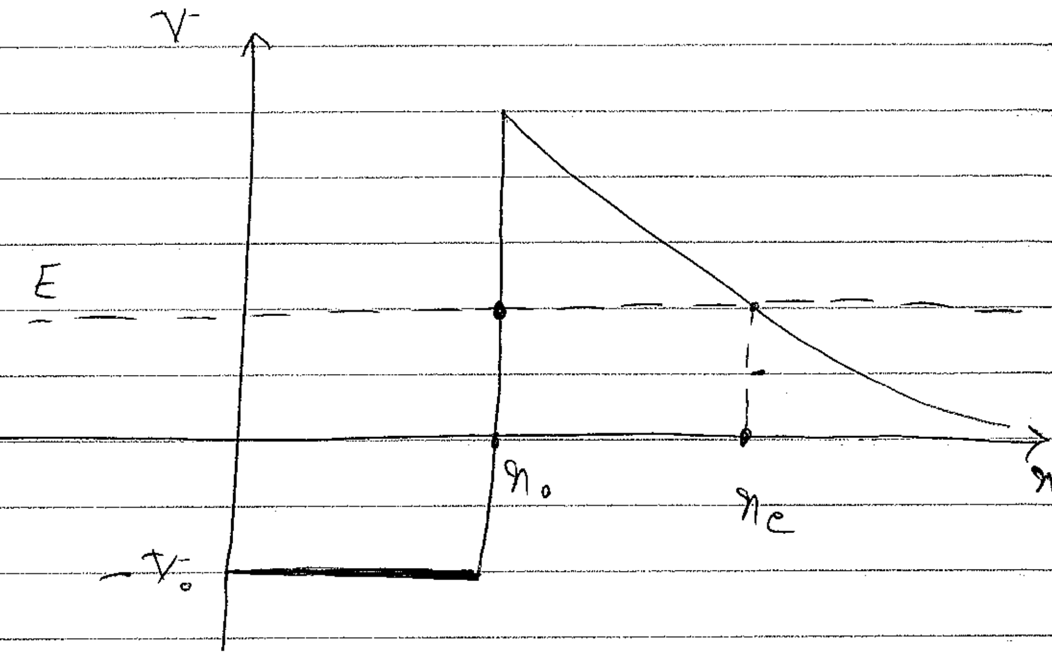
Such a well, with one infinite barrier, may have no bound states. In particular, if the depth of the well V_0 is not large enough, this will happen.

In the case of α particle the situation is similar. The bound states have negative energy in this case. The depth of potential $-V_0$ depends on the ratio of the number of protons and neutrons in the nucleus $\frac{N_p}{N_n}$. The larger $\frac{N_p}{N_n}$, the shallower the potential at $r<r_0$ is (because of the larger contribution of the repulsive Coulomb force).

Thus, nuclei with a large number of protons cannot support bound states of α particles. This is the reason why α decay happens and the α particle leaks out in such a case.

Now let us estimate the tunneling rate of an

α particle.



Consider an energy eigenstate with $E>0$. In the region $r_0 < r < r_e$ we have $E < V$. Therefore, using the WKB approximation, we find:

$$\psi(r_e) \simeq \psi(r_0) \sqrt{\frac{P(r_e)}{P(r_0)}} \; \exp\left[-\frac{1}{\hbar}\int_{r_0}^{r_e} \sqrt{2m\left(V(r)-E\right)}\; dr\right]$$

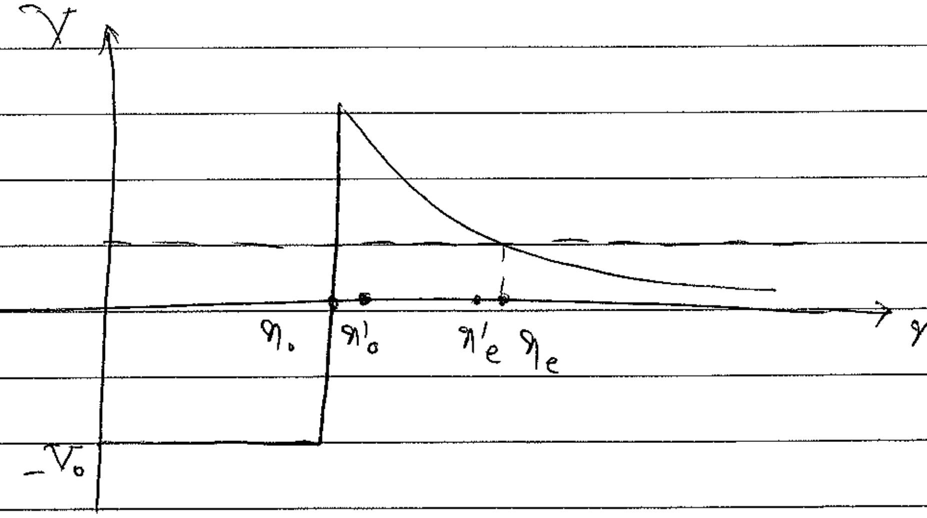
Note that we have ignored the exponentially growing term. This is reasonable if $r_e \gg r_0$, since in the limit that $r_e \to \infty$ the growing term vanishes.

The WKB approximation is not valid around r_0

and x_e because they are classical turning points

As we mentioned, the criterion for slow variation of potential is not satisfied very close to turning points.

Instead, we should choose points x'_0 and x'_e that are close to x_0 and x_e, respectively, and then we have a valid approximation:



$$\frac{\psi(x'_e)}{\psi(x'_0)} \simeq \sqrt{\frac{p(x'_e)}{p(x'_0)}} \exp\left[-\frac{1}{\hbar}\int_{x'_0}^{x'_e} \sqrt{2m(V_{(x)}-E)}\, dx\right]$$

Note, however, that the integrand is very small between x_0, x'_0 and between x'_e, x_e. Hence;

$$\int_{x'_0}^{x'_e} \sqrt{2m\,(V_{(x)} - E)}\; dx \simeq \int_{x_0}^{x_e} \sqrt{2m\,(V_{(x)} - E)}\; dx$$

Moreover, at a turning point $\frac{d^2\psi}{dx^2} = 0$. This implies that ψ varies slowly near a turning point since very close to it $\psi_{(x)} \simeq \psi_{(x_0)} + \frac{d\psi_{(x_0)}}{dx}(x - x_0)$. As a result:

$$\frac{\psi(x'_e)}{\psi_{(x'_0)}} \simeq \frac{\psi(x_e)}{\psi_{(x_0)}}$$

We therefore conclude that:

$$\frac{\psi_{(x_e)}}{\psi_{(x_0)}} \propto \exp\left[\frac{-1}{\hbar}\int_{x_0}^{x_e} \sqrt{2m\left(V_{(x)} - E\right)}\; dx\right]$$

And:

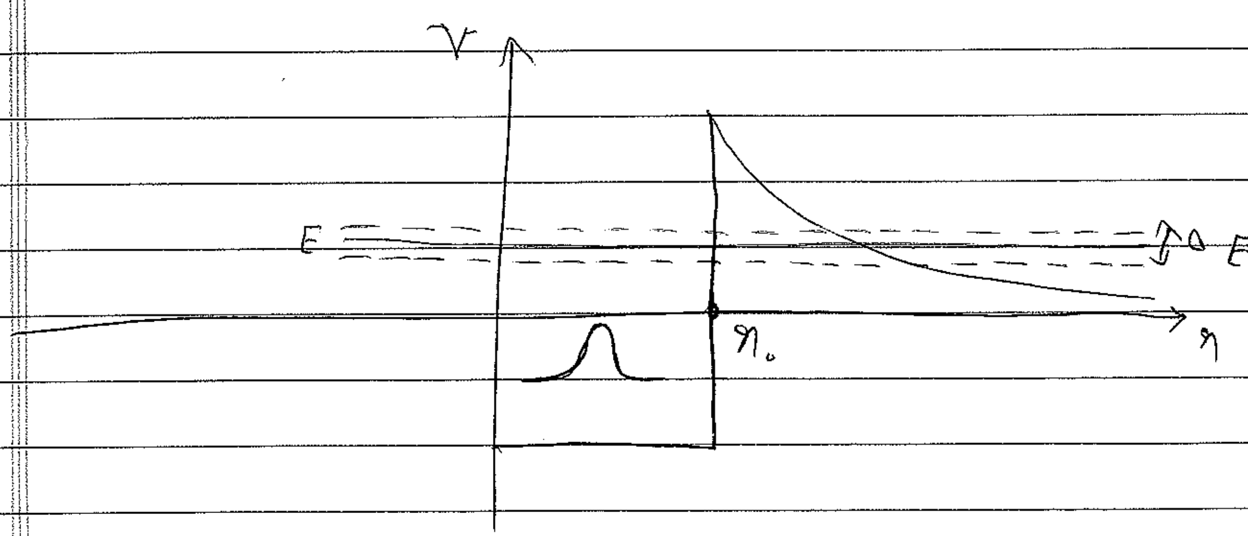
$$\frac{P(x_e)}{P_{(x_0)}} = \left|\frac{\psi(x_e)}{\psi_{(x_0)}}\right|^2 \propto \exp(-\gamma)$$

$P_{(x)}dx$ is the probability to find the particle between x and $x + dx$, and:

$$\gamma = \frac{2}{\hbar}\int_{x_0}^{x_e} \sqrt{2m\,(V_{(x)} - E)}\; dx$$

Now we would like to calculate the tunneling rate, i.e. the number of α particles that appear at r_e per unit time divided by the number of α particles inside the nucleus.

We note that an α particle confined within $0 < r < r_0$ is not an eigenstate but rather a superposition of eigenstates. Superposition of eigenstates in a narrow range ΔE around energy E yields a localized wavepacket:



The wavepacket moves back and forth in

the region between 0 and x_0. Each time that it is around x_0, there is a probability $\frac{P(x_e)}{P(x_0)} \propto e^{-\gamma}$ that it leaks outside and appears at x_e.

We have to find the fraction of time that the wavepacket spends around x_0. If $\Delta E \ll E$, it has a group velocity (spreading is insignificant);

$$v = \frac{p}{m} = \frac{\sqrt{2m(E+V_0)}}{m} = \sqrt{\frac{2(E+V_0)}{m}}$$

For $x < x_0$ we just have a particle in a box and $p^2 = 2m(E+V_0)$. The total distance travelled by the wavepacket in one second is v. The distance that it travels between two successive arrival at x_0 is $2x_0$ (from one end to the other and reverse). Thus the wavepacket spends $\left(\frac{2x_0}{v}\right)$ of a second around x_0. The tunneling rate R is then found to be,

$$R = \frac{v}{2r_0} e^{-\gamma} = \sqrt{\frac{E+V_0}{2mr_0^2}}\, e^{-\gamma}$$

The decay lifetime is:

$$\tau \equiv R^{-1} = \sqrt{\frac{2mr_0^2}{E+V_0}}\, e^{\gamma}$$

As m increases, γ will increase as well. Hence the tunneling rate decreases. This is expected physically: a heavier particle is less likely to climb up the barrier.

Also, for a smaller value of E, γ will increase. Thus the tunneling rate decreases. Note that for a smaller E the integrand $\sqrt{2m(V_{(r)} - E)}$ increases and the interval over which we take the integral $\int_{r_0}^{r_e} \sqrt{2m(V_{(r)} - E)}\, d_r$ increases. A less energetic particle is less likely to climb up the barrier, as expected physically.